Eigenvalues of absolute Cesàro matrices

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ABSTRACT

A complete description is given of the eigenvalues of the matrix A associated with absolute Cesàro summability of any positive order, where A is regarded as an operator on the Banach space l_p, with $1 \le p \le \infty$.

1. INTRODUCTION

For any real number $\alpha \ge 0$ and any given sequence $x = (x_0, x_1, \dots)$ of complex numbers consider the transformation defined by

$$(1) \qquad y_0 = x_0,\ y_n = \frac{1}{nA_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} k x_k.$$

This transformation is used in summability theory, whereby a series $\sum_{k=0}^\infty x_k$ is called summable $|C,\alpha|$, that is absolutely summable (C,α), if $\sum_{n=0}^\infty |y_n| < \infty$. The idea appears to have been introduced by Fekete [3] and has been studied and generalized by several authors ([1], [2], [4]).

In (1) we employ the usual notation, where for each real β we define

$$A_0^\beta = 1,\ A_n^\beta = (\beta+1)(\beta+2)\cdots(\beta+n)/n!$$

for $n \ge 1$.

In view of (1) we may regard summability $|C,\alpha|$ as arising from the infinite matrix $A = (a_{nk})$:

$$a_{00}=1, a_{0k}=0 \quad (k\geq 1),$$

$$a_{nk}=\frac{kA_{n-k}^{\alpha-1}}{nA_n^{\alpha}} \quad (0\leq k\leq n;\ n\geq 1)$$

and $a_{nk}=0$ $(k>n\geq 1)$.

Our object is to determine the point spectrum, that is the set of eigenvalues, of A acting as an operator on the Banach space l_p, where

$$l_p=\{x=(x_k):\ \sum_{k=0}^{\infty}|x_k|^p<\infty\}, \quad 1\leq p<\infty,$$

$$l_\infty=\{x=(x_k):\sup_k |x_k|<\infty\}.$$

There is also interest in the complete p-normed space

(2) $$l_p=\{x=(x_k):\|x\|=\sum_{k=0}^{\infty}|x_k|^p<\infty\}, \quad 0<p<1,$$

but as we shall show, if $\alpha>0$ and $0<p<1$ then A does not map l_p into itself. In (2) we have that $\|\cdot\|$ is a p-norm in the sense that $\|\lambda x\|=|\lambda|^p\|x\|$ for all complex λ and all $x\in l_p$; $\|x+y\|\leq\|x\|+\|y\|$ for all $x,y\in l_p$ $(0<p<1)$ and $\|x\|=0$ if and only if $x=0$.

For the point spectrum of A acting on l_p we use the notation $P\sigma(A,l_p)$, so that $\lambda\in P\sigma(A,l_p)$ if and only if there is a non-zero $x\in l_p$ such that $Ax=\lambda x$.

2. OPERATORS ON l_p

We now show that the matrix A which defines $|C,\alpha|$ summability acts as a bounded linear operator on l_p when $1\leq p\leq\infty$.

2.1. THEOREM. If $\alpha\geq 0$ and $1\leq p\leq\infty$ then $A\in B(l_p)$, that is A is a bounded linear operator on l_p.

PROOF. It is enough to show that $A\in B(l_1)$ and $A\in B(l_\infty)$, since it is known (see for example Maddox [6], page 174) that $B(l_1)\cap B(l_\infty)\subset B(l_p)$ for $1<p<\infty$.

Now it is well-known (Maddox [6]) that

(3) $$A\in B(l_\infty) \text{ if and only if } \sup_n \sum_{k=0}^{\infty}|a_{nk}|<\infty,$$

(4) $$A\in B(l_1) \text{ if and only if } \sup_k \sum_{n=0}^{\infty}|a_{nk}|<\infty.$$

In order to show (3) we use the following standard identity which is valid for any real α and β:

(5) $$\sum_{i=0}^{n} A_i^{\alpha}A_{n-i}^{\beta}=A_n^{\alpha+\beta+1}.$$

For $n=0$ the sum in (3) is equal to 1 and for $n\geq 1$ the sum is equal to

$$\sum_{k=0}^{n} kA_{n-k}^{\alpha-1}/nA_n^\alpha = \sum_{k=1}^{n} A_{k-1}^{1}A_{n-k}^{\alpha-1}/nA_n^\alpha$$

$$= \sum_{i=0}^{n-1} A_i^1 A_{n-1-i}^{\alpha-1}/nA_n^\alpha$$

$$= A_{n-1}^{\alpha+1}/nA_n^\alpha, \text{ by (5)},$$

$$= 1/(\alpha+1).$$

It follows from (3) that $A\in B(l_\infty)$.

To deal with (4) we may employ the following result of Chow [2],

LEMMA 1:

$$\text{(6)} \qquad \sum_{n=k}^{\infty} A_{n-k}^{\delta}/nA_n^\sigma = 1/kA_k^{\sigma-\delta-1}$$

where $k\geq 1$, $\sigma>-1$, $\sigma-\delta>0$.

For $k=0$ the sum in (4) is clearly equal to 1 and for $k\geq 1$ the sum is, by (6), also equal to 1, on taking $\delta=\alpha-1$ and $\sigma=\alpha$. This proves the theorem.

2.2. THEOREM. If $\alpha>0$ and $0<p<1$ then A does not map l_p into itself.

PROOF. Suppose, if possible, that A maps l_p into itself. Now apply the uniform boundedness principle to the sequence (q_r) of functionals on l_p as in (2), where

$$q_r(x) = \sum_{n=0}^{r} \Big| \sum_{k=0}^{n} a_{nk}x_k \Big|^p.$$

Then there is a positive constant M such that $q_r(x)\leq M\|x\|$ for all $r\geq 0$ and all $x\in l_p$. This result follows from an obvious modification of Theorem 11, page 114, of Maddox [6]. Hence, choosing $x=(0,0,\dots,1,0,0,\dots)$, the k-th. unit vector in l_p, we see that

$$\text{(7)} \qquad \sum_{n=0}^{\infty} |a_{nk}|^p \leq M, \text{ for all } k\geq 0.$$

By (7) and the definition of A it follows that

$$\text{(8)} \qquad k^p \sum_{n=k}^{\infty} (A_{n-k}^{\alpha-1})^p/(nA_n^\alpha)^p \leq M$$

for all $k\geq 1$.

Now for any $\beta>-1$ we have

$$A_n^\beta \sim n^\beta/\Gamma(\beta+1),$$

whence there is a positive number $c=c(\beta)$ with $A_n^\beta \leq cn^\beta$ for all $n\geq 1$. Hence, taking that part of the sum in (8) over the range $k\leq n\leq 2k$ we see that there is a positive constant H such that

$$(9) \qquad \sum_{i=0}^{k} (A_i^{\alpha-1})^p \leq Hk^{\alpha p}.$$

By a standard theorem on resultants (Hardy [5]) the sum in (9) is asymptotic to

$$ck^{(\alpha-1)p+1}$$

where c is a positive constant. It follows from (9) that (k^{1-p}) is bounded, which is contrary to the fact that $p<1$. This proves the theorem.

3. EIGENVALUES OF THE ABSOLUTE CESÀRO MATRIX

In the following theorem we determine the eigenvalues of the matrix A associated with summability $|C,\alpha|$, where $\alpha>0$, with A acting on l_p $(1\leq p\leq\infty)$.

3.1. THEOREM. Let $\alpha>0$. Then
(i) $P\sigma(A,l_p)=\{1\}$, for $1\leq p<\infty$.
(ii) $P\sigma(A,l_\infty)=\{1,1/(\alpha+1)\}$.

PROOF. Let $1\leq p\leq\infty$ and suppose $\lambda\in P\sigma(A,l_p)$. Then there exists $x\in l_\infty$ with $x\neq 0$ and $Ax=\lambda x$.

Suppose, if possible, that $x_0=0$ and $x_1=0$. We shall prove that this supposition leads to a contradiction. Since $x\neq 0$ we may choose the smallest $m\geq 2$ such that $x_m\neq 0$. Then for all $n\geq m$ we have

$$(10) \qquad \sum_{k=m}^{n} A_{n-k}^{\alpha-1} kx_k=\lambda nA_n^\alpha x_n,$$

whence $mx_m=\lambda mA_m^\alpha x_m$, so that $\lambda=1/A_m^\alpha$.

Next, putting $n=m+1$ in (10) and using the value of λ just found a short calculation shows that $x_{m+1}=mx_m=A_1^{m-1}x_m$.

We shall prove inductively that

$$(11) \qquad x_{m+i}=A_i^{m-1}x_m \text{ for all } i\geq 0.$$

Let us suppose that

$$x_{m+i}=A_i^{m-1}x_m \text{ for } 0\leq i\leq n-m.$$

Then (10) implies

$$A_m^\alpha \sum_{i-0}^{n\ m} A_{n-m+1-i}^{\alpha-1} mA_i^m x_m=(A_{n+1}^\alpha-A_m^\alpha)(n+1)x_{n+1}.$$

After some simplification this equation reduces to

$$x_{n+1}=\frac{n!A_m^\alpha mx_m}{(\alpha+1)\cdots(\alpha+m)(n-m+1)!}=A_{n+1-m}^{m-1}x_m$$

whence (11) holds. Then, since $m \geq 2$ and $x_m \neq 0$, we see by (11) that $|x_{m+i}| \to \infty$ as $i \to \infty$, contrary to the fact that $x \in l_\infty$. It follows that $x_0 \neq 0$ or $x_1 \neq 0$.

First let $1 \leq p < \infty$ and suppose, if possible, that $x_0 = 0$ but $x_1 \neq 0$. Then

$$x_1 = \lambda A_1^\alpha x_1 = \lambda(\alpha+1)x_1,$$

whence $\lambda = 1/(\alpha+1)$. A calculation then shows that $x_n = x_1$ for all $n \geq 1$, and since $x_1 \neq 0$ this contradicts the fact that $x \in l_p$. Hence when $1 \leq p < \infty$ and $\lambda \in P\sigma(A, l_p)$ we must have $x_0 \neq 0$. In this case it follows that $x_0 = \lambda x_0$, so $\lambda = 1$. Since $\alpha > 0$ it is immediate that $x_n = 0$ for all $n \geq 1$. Conversely, $1 \in P\sigma(A, l_p)$ when $1 \leq p < \infty$, since $x = (1, 0, 0, 0, \ldots)$ is a suitable eigenvector. This proves (i) of the theorem.

Finally, consider the case when $p = \infty$. If $x_0 = 0$ but $x_1 \neq 0$ then $\lambda = 1/(\alpha+1)$ and $x_n = x_1$ for all $n \geq 1$ and $x = (0, 1, 1, 1, \ldots)$ is a suitable eigenvector.

But if $x_0 \neq 0$ then $\lambda = 1$ and $x_n = 0$ for all $n \geq 1$ and $x = (1, 0, 0, \ldots)$ is an eigenvector. Thus the eigenvalues are 1 and $1/(\alpha+1)$, so (ii) holds and the proof is complete.

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